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for Certain Classes of
n-Person Games

by

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STABILITY AND STRUCTURAL THEOREMS FOR
CERTAIN CLASSES OF n-PERSON GAMES

R. Duncan Luce

1. Introduction[1]

In the von Neumann and Morgenstern (4) formulation of game theory, the n-person case with a transferable utility is reduced, when $n>2$, by means of the 2-person theory to the study of certain types of real-valued set functions which they have called characteristic functions. In words, a characteristic function assigns to each subset of players a number which represents (in certain units) the "strength" of that set of players if they cooperate as a coalition and if they are opposed by all the remaining players cooperating as a coalition. Since in some situations the opposition may not be unified into a single coalition, the characteristic function must be considered to give a conservative estimate of coalition strength.

The units in which the characteristic function is measured are assumed to be those of an extra-player commodity which acts like money, i.e., it is infinitely divisible and freely transferable among the players. It need not, however, be ordinary money or be simply related to it, but it is the quantity in which the players are "paid" at the end of the game.

Mathematically, an n-person game (in characteristic function form), $n \geq 3$, is a pair (I_n, v), where I_n is the set of n players - which for all purposes can be taken to be a labeling of the players by the first n integers - and where v, the characteristic function, is a real valued set function defined for all subsets of I_n which satisfies

1. If the reader is familiar with either (1) or (2) he should omit the introduction through the definition of k-stability.

$$\text{i. } v(\phi) = 0, \text{ where } \phi \text{ is the null set,}$$

and

$$\text{ii. if R and S are disjoint subsets of } I_n, \quad v(R \cup S) \geq v(R) + v(S).$$

The first condition simply assigns the value zero to the coalition having no members and the second is a way of saying that any whole of players is at least as "strong" as any sum of disjoint parts.

It turns out that the theories so far developed are invariant over certain equivalence classes of characteristic functions and so it is sufficient to isolate one representative function from each of the classes. We shall not repeat the argument here; it may be found in (2,3). One particularly convenient representative can be shown to be the unique characteristic function m in each class except one (see below) which in addition to i and ii above satisfies

$$\text{iii. } m(\{i\}) = 0, \text{ for } i \in I_n,$$

and

$$\text{iv. } m(I_n) = 1.$$

This function is known as the <u>0,1-reduced form</u>, and it can be shown that the 0,1-reduced form of the class generated from a given characteristic function v is

$$m(S) = \frac{v(S) - \sum_{i \in S} v(\{i\})}{v(I_n) - \sum_{i \in I_n} v(\{i\})},$$

except in the case $v(I_n) = \sum_{i \in I_n} v(\{i\})$, in which class there does not exist a 0,1-reduced form. A game of this exceptional type is called <u>inessential</u>, and it is substantially the same as a probability measure over the set of players. It is easy to see that in an inessential game there is no gain achieved by forming coalitions and so the coalition theory is trivial; hence these games are excluded from further study. Any game which is not inessential is generally called <u>essential</u>; however, in this paper we shall use the word "game" to mean essential game unless it is prefixed by "inessential."

An important class of games - though it will play only a minor role here - is that for which[2]

$$\text{v. } v(S) + v(-S) = 1, \text{ for every } S \subset I_n.$$

Such a game is called <u>constant-sum</u>.

In addition to the notion of coalition strength as embodied in the characteristic function, von Neumann and Morgenstern introduced the concept of payments received by the players. This payment includes the payments from the game plus any side payments resulting from coalition participation and it is measured in the units of the extra player commodity mentioned earlier. If we denote the payment received by player i by x_i, then it is assumed that

$$\text{i. } \sum_{i \in I_n} x_i = v(I_n),$$

and

$$\text{ii. } x_i \geq v(\{i\}), \text{ for } i \in I_n.$$

Such an n-tuple, called an <u>imputation</u> of the game, assumes that the players - "rational" players - will divide up the value of the game for the set of all players and that each rational player will accept no arrangement which gives him less than he can assure himself were he to play alone with all of the other players in a coalition opposing him. If we assume the characteristic function is in 0,1-reduced form, then any imputation is simply a probability distribution over the set of players, and conversely a distribution is an imputation.

The von Neumann-Morgenstern theory of "solutions" (3,4) attempts, using only these concepts, to characterize sets of imputations which, with

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2. If $S \subset I_n$, then $-S$ denotes the set $I_n - S$.

respect to the given characteristic function, achieve a certain inner stability for "rational" players. On the other hand, the theory we shall discuss here, which was introduced earlier by the author (1), requires one further notion. Any system τ of non-overlapping proper subsets of I_n which exhaust I_n is called a <u>coalition structure</u>. We shall take the point of view that the outcome of a game will be described by a pair (X,τ), where X is an imputation and τ is the corresponding coalition structure. One task of a theory is to describe which pairs will be in a state of equilibrium relative to the given characteristic functions. We have mentioned in (2) other problems which may be posed, but in this paper we shall be concerned only with equilibrium behavior.

The basic idea we shall employ is that even when a pair (X,τ) has been accepted, "rational" players will be shopping around for revisions of the coalition structure which will benefit them. The pair (X,τ) will be in a state of equilibrium when no improvement is assured by any of the possible changes, and hence there will be no motive for a change. It turns out, however, that no very rich theory will result unless it is assumed that the players may only consider modifications of τ which are not too extreme. Such an assumption is not implausible for at least two reasons: it is a recognition that, at least with real people, overly complex changes of alliance cannot be effected, and that in some economic situations very complicated changes are very expensive. In any case, we shall suppose that only certain changes from τ can be considered by the players, and that if any admissible change is certain to be profitable then the arrangement (X,τ) will be disrupted. It is quite possible to define the notion quite generally, as in (2); however, in this paper we shall examine only certain special cases - those defined in (1) - and so the definition will be given only for that case.

Let k be an integer with $0 \leq k \leq n-2$ and let τ be a coalition structure. A subset $S \subset I_n$ is called a <u>k-critical</u> coalition of τ if there exists a $T\varepsilon\tau$ such that[3]

$$|(S-T)\cup(T-S)| \leq k.$$

A pair (X,τ) is said to be <u>k-stable</u> if for every k-critical coalition S of

$$v(S) \leq \sum_{i\varepsilon S} x_i$$

and if for every $i\varepsilon T$, where $T\varepsilon\tau$ and $|T| > 1$, $x_i > 0$.

The first, and more important condition, states that for any S which does not differ by more than k elements from a member of τ the characteristic function value of S shall not exceed the total already agreed upon in X for the players who would form S. In other words, there is not a positive incentive for the coalition S to form. Since this is true of all S's which may be considered, there will be no tendency for (X,τ) to be destroyed. The second condition simply reflects the intuition that a player will not participate in a non-trivial coalition unless he receives more than he could assure himself when playing alone in the most adverse situation. This second condition seems only to serve the role of reducing the number of k-stable pairs, for we have yet to find a case where a game does not have a k-stable pair if the condition is assumed and that it does when the condition is dropped.

A game is called <u>k-stable</u> if there is at least one k-stable pair, otherwise it is called <u>k-unstable</u>. It is easy to see that if $k<k'$, then k-unstable implies k'-unstable and that k'-stable implies k-stable.

A particularly interesting case of k-stability is k = 1, for this case is, in a way, the borderline between stable and unstable games. If some change is allowed every coalition of τ, then the least possible is

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3. If $S \subset I_n$, then $|S|$ denotes the number of elements in S.

with $k = 1$, and so a game which is 1-unstable is, in this sense, inherently unstable. Thus, even when it is not possible at present to obtain full stability results for a class of games, it is still interesting to separate the 1-unstable ones from those which are 1-stable.

In (1) we began to study the conditions for k-stability in certain general classes of games. There we covered the 3- and 4-person constant-sum games, simple games, and negative games. We continue this program here and present similar results for symmetric and quota games. This is the content of the first part of the paper. In the second part, we shall present some structural theorems for simple quota games and for two other classes of simple games which are closely related to the non-constant-sum simple quota games. These are not stability theorems as such, but they are indirectly a product of our consideration of the stability problem.

I. ON THE STABILITY OF SYMMETRIC AND QUOTA GAMES

2. Discretely Stable Games

In the previous section we emphasized the important dichotomy between 1-stable and 1-unstable games. It appears desirable, for reasons which will become apparent, to refine this into a trichotomy by dividing the 1-stable games into two classes. Let us denote the special coalition structure $[\{1\},\{2\},\ldots \{n\}]$ by Δ_n; it is the case of pure competition within the universe of the n players. We shall call a 1-stable game which has a 1-stable pair of the form (X, Δ_n) discretely stable, and those 1-stable games with no such 1-stable pair will be called non-discretely stable. It is trivial to see that discretely stable games exist, but an example of a non-discretely stable one is needed. Consider any (I_n, m) having the following properties: there exists a set T such that

i. $|T| = \frac{n}{2} = t,$

ii. for $i,j \varepsilon T$ or $i,j \varepsilon -T$, $m(\{i,j\}) > 2/n,$

iii. for $i,j \varepsilon T$ and $k \varepsilon -T$ or $i \varepsilon T$ and $j,k \varepsilon -T$,

$m(\{i,j,k\}) \leq 3/n.$

It is not difficult to see that such games exist. Now suppose a pair (X, Δ_n) is 1-stable, then for $i,j \varepsilon T$ and $i,j \varepsilon -T$,

$$x_i + x_j \geq m(\{i,j\}) > 2/n.$$

If we sum over all possible pairs in T,

$$(t-1) \sum_{i \varepsilon T} x_i > \frac{t(t-1)}{2} \frac{2}{n},$$

and so

$$\sum_{i \varepsilon T} x_i > t/n = 1/2.$$

Similarly, $\sum_{i \in -T} x_i > 1/2$, and a contradiction results. On the other hand, if we relabel the players so that $T = \{1,2,\ldots,t\}$, then the pair $(\|1/n\|, [\{1,t+1\},\{2,t+2\}, \ldots,\{t,n\}])$ is 1-stable. To show this we need only consider two distinct types of 1-critical coalitions:

$$m(\{1,t+1,t+j\}) \leq 3/n = x_1 + x_{t+1} + x_{t+j},$$

$$m(\{1,t+1,j\}) \leq 3/n = x_1 + x_{t+1} + x_j,$$

and so the game is 1-stable and hence non-discretely stable.

This concept of discrete stability is actually a generalization of the notion of a quota game without a weak player. Shapley (5) has called a game for which there exists an n-tuple $Q = \|q_i\|$ such that

$$\text{i. } \sum_{i \in I_n} q_i = 1,$$

and

$$\text{ii. } m(\{i,j\}) = q_i + q_j, \quad i \neq j,$$

a <u>quota game</u>, where Q is called the <u>quota</u>. A player i is called <u>weak</u> if $q_i < 0$. Since $m(\ i,j\ ) \geq 0$, it is clear that there is at most one weak player. Now, compare this with the fact in a discretely stable game there exists an n-tuple X such that

$$\text{i. } \sum_{i \in I_n} x_i = 1,$$

$$\text{ii. } m(\{i,j\}) \leq x_i + x_j, \quad i \neq j,$$

$$\text{iii. } x_i \geq 0.$$

The notion of discrete stability is of interest, first, because all 1-stable simple, negative, symmetric, or quota games are discretely stable. This is obvious for simple and negative games from theorems 4 and 7 of (1), and for symmetric and quota games it follows from theorems 1 and 2 below. We must therefore conclude that the special types of games which have been given detailed study do not give us any insight into the phenomena of non-discrete stability.

A second reason to concern ourselves with this classification is a property of (X, Δ_n) k-stable pairs which suggests that they may be expected to arise only rarely in empirical situations. The definition of k-stability implicitly supposes a dynamic model in which a change from a (non-stable) pair (X,τ) occurs if there is a k-critical coalition S of τ such that $m(S) < \sum_{i\varepsilon S} x_i$. From a given τ only certain other coalition structures can be reached by means of changes employing only k-critical coalitions, i.e., from a given τ there will be coalition structures which are inaccessible. Indeed, it is not impossible that there are some structures which are not accessible from any other pair. Formally, a coalition structure τ is <u>k-inaccessible</u> if for every pair (X,τ') such that

i. $m(T) \leq \sum_{i\varepsilon T} x_i$, $T\varepsilon\tau'$

and ii. the coalitions of τ are k-critical coalitions of τ',

then iii. $m(S) \leq \sum_{i\varepsilon S} x_i$ for every $S\varepsilon\tau$.

Within the framework of this dynamic model a k-stable pair (X,τ), where τ is k-inaccessible, can never arise in a trial and error fashion from other tentatively accepted pairs, but rather it must be agreed upon at the very beginning of the coalition formation process. The latter occurrence seems to be rather unlikely in practice, and so we must expect such k-stable pairs to play a special role. Clearly a sufficient condition for τ to be k-inaccessible is that $m(T) = 0$ for every $T\varepsilon\tau$ (in the conventional language of game theory, T is losing), and so we see that Δ_n is k-inaccessible.

It appears from these observations that the full significance of the following stability theorems will only become understood when a complete dynamic theory is presented and one can answer such questions as the probability that a k-stable pair will arise given the starting pair, or a distribution over starting pairs, of the players.

3. Symmetric Games

One quite general and important class of games which has been studied in the literature are those in which the characteristic function depends only on the size of a coalition, that is,

$$m(T) = m(|T|).$$

Such games are called symmetric.

Theorem 1. *A symmetric game with characteristic function $m(i)$ is k-stable if and only if $m(i) \leq i/n$ for $0 \leq i \leq k+1$.*

Proof. It is clear that $(\|1/n\|, \Delta_n)$ is k-stable if the condition is met.

Conversely, suppose (X, τ) is k-stable and that $m(k+1) > (k+1)/n$. Consider any positive integer a such that $a(k+1) \leq n$. Since we may partition any coalition of $a(k+1)$ elements into a disjoint coalitions of $k+1$ elements,

$$m[a(k+1)] \geq am(k+1) > a(k+1)/n.$$

For any $T_i \varepsilon \tau$ it is clear that we may write

$$|T_i| = a\ (k+1) + b_i,$$

where a_i and b_i are integers such that

$$0 < a_i(k+1) < n \quad \text{and} \quad -k \leq b_i \leq k.$$

Let us denote the quantity $\sum_{j\varepsilon T_i} x_j$ by d_i, and then we consider three cases.

1. $b_i = 0$. From the condition of k-stability we have

$$d_i \geq m(|T_i|) = m[a_i(k+1)] > a_i(k+1)/n = |T_i|/n.$$

2. $b_i < 0$. We first show that it is always possible to find a set B_i such that

$$B_i \subset -T_i, \ |B_i| = |b_i|, \text{ and} \sum_{j\varepsilon B_i} x_j \le \frac{(1-d_i)|b_i|}{n - |T_i|} .$$

If this were not the case, then we would have to assume that for the $\binom{n - |T_i|}{|b_i|}$ coalitions B_i meeting the first two conditions that

$\sum_{j\varepsilon B_i} x_j > \frac{(1-d_i)|b_i|}{n - |T_i|}$. Observe that each $j\varepsilon -T_i$ appears in exactly

$\binom{n - |T_i| - 1}{|b_i| - 1}$ of these sets, and so if we sum over all of them we

obtain
$$\sum_{B_i}\sum_{j\varepsilon B_i} x_j = \binom{n - |T_i| - 1}{|b_i| - 1} \sum_{j\varepsilon -T_i} x_j = \binom{n - |T_i| - 1}{|b_i| - 1} (1 - d_i)$$

$$> \binom{n - |T_i|}{|b_i|} \frac{(1-d_i)|b_i|}{n - |T_i|} = \binom{n - |T_i| - 1}{|b_i| - 1} (1 - d_i) ,$$

which contradiction establishes the existence of a B_i meeting the three conditions. Since $|B_i| = |b_i| \le k$, $T_i \cup B_i$ is a k-critical coalition of τ and so

$$\begin{aligned} d_i + \frac{(1 - d_i)|b_i|}{n - |T_i|} &\ge \sum_{j\varepsilon T_i} x_j + \sum_{j\varepsilon B_i} x_j \\ &\ge m(|T_i| + |b_i|) \\ &= m[a_i(k+1)] \\ &> a_i(k+1)/n. \end{aligned}$$

Thus,

$$\frac{d_i(n-|T_i|-|b_i|) + |b_i|}{n - |T_i|} > a_i(k+1)/n,$$

or $$d_i > \frac{(n-|T_i|)a_i(k+1) - n|b_i|}{n(n-|T_i| - |b_i|)} = \frac{(n-|T_i|)(|T_i| + |b_i|) - n|b_i|}{n(n-|T_i| - |b_i|)}$$

$$= |T_i|/n.$$

3. $b_i > 0$. We first show that it is always possible to find a set B_i such that

$B_i \subset T_i$, $|B_i| = b_i$, and $\sum_{j \varepsilon B_i} x_j \geq d_i b_i/|T_i|$. If this were not the case, then we may sum over all $\binom{|T_i|}{b_i}$ sets B_i satisfying the first two conditions, and we obtain

$$\sum_{B_i} \sum_{j \varepsilon B_i} x_j = \binom{|T_i| - 1}{|b_i| - 1} d_i < \binom{|T_i|}{b_i} \frac{d_i b_i}{|T_i|} = \binom{|T_i| - 1}{b_i - 1} d_i,$$

which is a contradiction. Observe that for any B_i satisfying the three conditions, $T_i - B_i$ is a k-critical coalition of τ, hence

$$\begin{aligned} d_i - d_i b_i/|T_i| &\geq \sum_{j \varepsilon T_i} x_j - \sum_{j \varepsilon B_i} x_j \\ &\geq m[a_i(k+1)] \\ &> a_i(k+1)/n \\ &= (|T_i| - b_i)/n. \end{aligned}$$

Thus, $$d_i > |T_i|/n.$$

We have therefore shown that for every $T_i \varepsilon \tau$, $\sum_{j \varepsilon T_i} x_j > |T_i|/n$, and so

$$1 = \sum_{j \varepsilon T_n} x_j = \sum_{T_i} \sum_{j \varepsilon T_i} x_j > \sum_{T_i} |T_i|/n = 1,$$

which is impossible and so the pair is not k-stable, and the theorem is proved.

In (1) we defined a game to be <u>negative</u> if $m(T) \leq |T|/n$ for all $T \subset I_n$.

<u>Corollary 1.</u> <u>A symmetric game is (n-2)-stable if and only if it is negative.</u>

Proof. The theorem and the definition of a negative game.

<u>Corollary 2.</u> <u>If a symmetric game is 1-stable it is discretely stable.</u>

Proof. Trivial.

4. Quota Games

It will be recalled (section 2) that Shapley (5) defined a <u>quota game</u> to be one such that there exists an n-tuple $Q = \|q_i\|$, called the <u>quota</u>, such that

$$\text{i.}\quad \sum_{i \in I_n} q_i = 1,$$

and

$$\text{ii.}\quad m(\{i,j\}) = q_i + q_j,\ i,j \in I_n.$$

A player i is called <u>weak</u> if $q_i < 0$, and we noted that there is at most one weak player.

<u>Theorem 2.</u> <u>A quota game is 1-stable if and only if there is no weak player.</u>

Proof. If there is no weak player, then clearly the pair (Q, Δ_n) is 1-stable.

Conversely, suppose there is a weak player, which by relabeling we may take to be n, and let (X, τ) be a 1-stable pair. Label the coalitions $T_1, \ldots, T_t$ of τ so that $n \in T_t$. For any $T_i \in \tau$, the 1-stability requirement implies

$$m(T_i) \leq \sum_{j \in T_i} x_j.$$

Now, if $|T_i|$ is even, then T_i can be partitioned into $|T_i|/2$ non-overlapping two element coalitions, each of which has the value $m(\{i,j\}) = q_i + q_j$.

Thus, $$\sum_{j\epsilon T_i} x_j \geq m(T_i) \geq \sum_{j\epsilon T_i} q_j.$$

If $|T_i| > 1$ and is odd, then for every $k\epsilon T_i$, $|T_i-\{k\}|$ is even, and so by the same argument

$$\sum_{j\epsilon T_i-\{k\}} x_j \geq \sum_{j\epsilon T_i-\{k\}} q_j.$$

Summing over all $k\epsilon T_i$,

$$\sum_{k\epsilon T_i} \sum_{j\epsilon T_i-\{k\}} x_j = (|T_i|-1) \sum_{j\epsilon T_i} x_j \geq \sum_{k\epsilon T_i} \sum_{j\epsilon T_i-\{k\}} q_j = (|T_i|-1) \sum_{j\epsilon T_i} q_j,$$

hence $$\sum_{j\epsilon T_i} x_j \geq \sum_{j\epsilon T_i} q_j.$$

If $|T_i| = 1$, let $T_i = \{i\}$, and then for any $k\epsilon-\{i\}$, $\{i,k\}$ is a 1-critical coalition and so

$$x_i + x_k \geq m(\{i,k\}) = q_i + q_k.$$

Summing over all $k\epsilon-\{i\}$,

$$(n-2)x_i + \sum_{k\epsilon I_n} x_k \geq (n-2)q_i + \sum_{k\epsilon I_n} q_k.$$

But $\sum_{k\epsilon I_n} x_k = 1 = \sum_{k\epsilon I_n} q_k$; so with $n \geq 3$,

$$x_i \geq q_i.$$

Since these inequalities hold for all $T_i\epsilon\tau$ and since $\sum_{k\epsilon I_n} x_k = \sum_{k\epsilon I_n} q_k$,

the equalities $$\sum_{j\epsilon T_i} x_j = \sum_{j\epsilon T_i} q_j = m(T_i), \text{ if } |T_i| \text{ is even}$$

$$\sum_{j\epsilon T_i} x_j = \sum_{j\epsilon T_i} q_j \qquad , \text{ if } |T_i| \text{ is odd}$$

must hold.

Next we show that if n is weak and $n \varepsilon T_t$, $|T_t|$ is even. Suppose, on the contrary, $|T_t|$ is odd. If $|T_t|>1$, then by the partitioning argument

$$m(T_t) \geq m(T_t-\{n\}) \geq \sum_{j \varepsilon T_t-\{n\}} q_j .$$

But we know that $\sum_{j \varepsilon T_t} q_j = \sum_{j \varepsilon T_t} x_j$, and since n is weak, $q_n<0$, so

$$m(T_t) \geq \sum_{j \varepsilon T_t} q_j - q_n > \sum_{j \varepsilon T_t} x_j ,$$

which violates the 1-stability assumption. If $|T_t| = 1$, then $T_t = \{n\}$ and we have shown above that $x_n = q_n < 0$, which is impossible. Thus $|T_t|$ is even.

It is clear that in $-T_t$ there is at least one k such that $q_k \geq x_k$. Consider the 1-critical coalition $T_t \cup \{k\}$. Since $|T_t|$ is even, so is $|(T_t \cup \{k\})-\{n\}|$, and so we may partition that coalition into non-overlapping two element coalitions:

$$m(T_t \cup \{k\}) \geq m[(T_t \cup \{k\})-n] \geq \sum_{i \varepsilon T_t} q_i + q_k - q_n .$$

But $q_n<0$ and $q_k \geq x_k$, so

$$m(T_t \cup \{k\}) > \sum_{i \varepsilon T_t} x_i + x_k + 0 = \sum_{i \varepsilon T_t \cup \{k\}} x_i ,$$

which violates the assumption that (X,τ) is 1-stable. Thus, we must conclude that there is no weak player.

<u>Corollary 1.</u> <u>All quota games with an odd number of players are 1-stable.</u>

Proof. If n is odd then there is no weak player, since if $q_n<0$,

$$m(I_n-\{n\}) \geq \sum_{i \varepsilon I_n-\{n\}} q_i = \sum_{i \varepsilon I_n} q_i - q_n > 1,$$

which is impossible. Thus the theorem implies a quota game is 1-stable when n is odd.

<u>Corollary 2.</u> <u>If a quota game is 1-stable it is discretely stable.</u>

Proof. Trivial.

<u>Theorem 3.</u> <u>Let</u> (I_n, m) <u>be a k-stable quota game and let</u> (X,τ) <u>be a k-stable pair.</u> <u>If</u> n <u>is odd or if</u> n <u>is even and</u> $k \geq 2$, <u>then</u> $X = Q$. <u>If</u> n <u>is even and</u> $k = 1$, <u>then either</u> $X = Q$ <u>or</u> $|T|$ <u>is even and</u> $m(T) = \sum_{i\epsilon T} q_i = \sum_{i\epsilon T} x_i$ <u>for every</u> $T\epsilon\tau$. <u>There are quota games (both constant-sum and non-constant-sum) with</u> n <u>even and</u> $k = 1$ <u>in which</u> $X \neq Q$.

Proof. Suppose (X,τ), where $\tau = (T_1,\ldots,T_t)$, is 1-stable and that for some r, $x_r \neq q_r$. From the proof of theorem 2 we know that for each $T_i\epsilon\tau$, $\sum_{j\epsilon T_i} x_j = \sum_{j\epsilon T_i} q_j$. It follows, therefore, that in some T_i, say T_t, there exist r and s such that $x_r > q_r$ and $x_s < q_s$. Now suppose that for $i \neq t$, $|T_i|$ is odd, then $T_i \cup \{s\}$ has an even number of elements and is 1-critical, so

$$\sum_{j\epsilon T_i \cup \{s\}} x_j \geq m(T_i \cup \{s\}) \geq \sum_{j\epsilon T_i \cup \{s\}} q_j = \sum_{j\epsilon T_i} q_j + q_s > \sum_{j\epsilon T_i \cup \{s\}} x_j,$$

which is impossible. Thus $|T_i|$ is even. If n is even, then so is $|T_t|$. Suppose n, and therefore $|T_t|$, is odd. Since we know that if $T_t = \{r\}$, $q_r = x_r$, it follows that $|T_t| > 1$. Since $|T_t - \{r\}|$ is even,

$$m(T_t - \{r\}) \geq \sum_{j\epsilon T_t - \{r\}} q_j > \sum_{j\epsilon T_t - \{r\}} x_j,$$

which is impossible. Thus, if (X,τ) is 1-stable either $X=Q$ or $|T|$ is even for $T\epsilon\tau$. Since any k-stable pair is also 1-stable, the conclusion holds for k-stable pairs. If $|T|$ is even we know from the proof of theorem 2 that $m(T) = \sum_{i\epsilon T} q_i = \sum_{i\epsilon T} x_i$.

Next, let us assume that n is even and $k \geq 2$, and suppose (X,τ) is k-stable and $X \neq Q$. Thus there exists $r\varepsilon T_i$, for some i, such that $x_r > q_r$, and for any $j \neq i$, there exists $s\varepsilon T_j$ such that $x_s \leq q_s$. Consider $(T_i-\{r\})\cup\{s\}$ which is k-critical for $k \geq 2$ and which has an even number of elements since T_i does. Thus,

$$m[(T_i-\{r\})\cup\{s\}] \geq \sum_{j\varepsilon T_i} q_j - q_r + q_s$$

$$> \sum_{j\varepsilon(T_i-\{r\})\cup\{s\}} x_j ,$$

which is impossible. Thus, $X = Q$.

The following is an example of a non-constant-sum symmetric quota game in which (X,τ) is 1-stable and $X \neq Q$:

$$n = 6,\ q_i = 1/6,\ m(2) = 4/12,\ m(3) = 5/12,\ m(4) = 8/12,\ m(5) = m(6) = 1.$$

It is easy to show that

$$(\|1/12,3/12,2/12,2/12,2/12,2/12\|,\ [\ 1,2\ ,\ 3,4\ ,\ 5,6\ ])$$

is 1-stable. This example is readily modified into a constant-sum example if $m(3)$ is altered as follows:

$$m(\{1,2,3\}) = m(\{1,5,6\}) = 5/12$$

$$m(\{2,5,6\}) = m(\{2,3,4\}) = 7/12$$

$$m(\{i,j,k\}) = 6/12 \text{ for all other } \{i,j,k\}.$$

The same pair is 1-stable.

<u>Theorem 4.</u> <u>Let (I_n,m) be a quota game without a weak player.</u> <u>A sufficient, but not a necessary, condition for the game to be k-stable,</u> $k \geq 2$, <u>is that</u>

$$m(T) \leq \sum_{i\varepsilon T} q_i \qquad (1)$$

<u>hold for all</u> T <u>such that</u> $|T| \leq k + 1$. <u>A necessary, but not a sufficient, condition for it to be k-stable is that equation 1 hold for all</u> T <u>such that</u>

$|T| \leq k$. Suppose the game is k-stable and there exists a set T such that $|T| = k + 1$ and equation 1 is violated, and let τ be the coalition structure of any k-stable pair. If T_i is any coalition of τ which intersects T, then $|T_i \cap T| = 1$ and $|T_i|$ is even.

Proof. Sufficiency: It is obvious that (Q, Δ_n) is k-stable if equation 1 holds for all T such that $|T| \leq k + 1$. To show that this condition is not necessary, consider the following game:

$n = 6$, $q_i = 1/6$

$m(S) = |S|/6$ for all S such that $|S| = 2,3,4,6$ except $\{1,3,5\}$ and $\{2,4,6\}$

$= 1$ for all S such that $|S| = 5$

$m(\{1,3,5\}) = 4/6$

$m(\{2,4,6\}) = 2/6$.

Let $\tau = [\{1,2\},\{3,4\},\{5,6\}]$ and then it is easy to see that (Q,τ) is 2-stable since any 2-critical coalition contains at most four elements and if it contains $\{1,3,5\}$ it must contain four. But equation 1 is violated for $T = \{1,3,5\}$.

Necessity: If the game is k-stable, $k \geq 2$, then by theorem 3 any k-stable pair is of the form (Q,τ). Suppose that T is a set such that $|T| \leq k + 1$ and $m(T) > \sum_{i \in T} q_i$. Let T_i be any coalition of τ which intersects T. First, $T_i - T \neq 0$, since if $T_i \subset T$, then $|T-T_i| \leq k$ because $|T| \leq k + 1$. In that case T is a k-critical coalition of τ and the hypothesis of k-stability is violated. Second, $T_i \cup T \neq I_n$, for if it were $= I_n$, then for any $s \in I_n$, $(T_i \cup T)-\{s\}$ is a k-critical coalition of τ. We may make $|(T_i-T)-\{s\}|$ even by choosing either $s \in T_i-T$ or $s \in -(T_i-T)$. Thus,

$$\begin{aligned} m[(T_i \cup T)-\{s\}] &= m\Big(T \cup [(T_i-T)-\{s\}]\Big) \\ &\geq m(T) + m[(T_i-T)-\{s\}] \\ &> \sum_{j\varepsilon T} q_j + \sum_{j\varepsilon (T_i-T)-\{s\}} q_j \\ &= \sum_{j\varepsilon (T_i \cup T)-\{s\}} q_j, \end{aligned}$$

which violates the k-stability condition. Thus we know that $T_i-T \neq 0$ and $T_i \cup T \neq I_n$. Suppose that $|T_i \cap T| > 1$. By choosing an element $s\varepsilon T_i-T$ or in $-(T_i \cup T)$, we may make $|(T_i-T) \cup \{s\}|$ even. Observe that since $|T_i \cap T| > 1$,

$$|(T \cup T_i \cup \{1\})-T_i| = |(T \cup \{1\})-(T \cap T_i)| \leq k,$$

and so $T \cup T_i \cup \{1\}$ is a k-critical coalition of τ. Then,

$$\begin{aligned} m(T \cup T_i \cup \{s\}) &= m\Big(T \cup [(T_i \cup \{s\})-T]\Big) \\ &= m[T \cup (T_i-T) \cup \{s\}] \\ &\geq m(T) + m[(T_i-T) \cup \{s\}] \\ &> \sum_{j\varepsilon T} q_j + \sum_{j\varepsilon (T_i-T) \cup \{s\}} q_j \\ &= \sum_{j\varepsilon T \cup T_i \cup \{s\}} q_j, \end{aligned}$$

and so we must conclude that $|T_i \cap T| = 1$.

Next, suppose that $|T_i|$ is odd, then $|T_i-T|$ is even and $T \cup T_i$ is k-critical, so

$$m(T \cup T_i) \geq m(T) + m(T_i-T) > \sum_{j\varepsilon T \cup T_i} q_j,$$

hence $|T_i|$ is even. Finally, if we suppose $|T| \leq k$, then for $s\varepsilon-(T \cup T_i)$

$$|(T \cup T_i \cup \{s\})-T_i| = |(T \cup \{s\})-(T \cap T_i)| \leq k,$$

so $T \cup T_i \cup \{s\}$ is k-critical. Since $|T_i|$ is even, $|(T_i-T) \cup \{s\}|$ is even, and the same argument as above leads to a contradiction. Thus, we must conclude that equation 1 holds for all T with $|T| \leq k$, and if it fails for some T with $|T| = k + 1$, then for any $T_i \epsilon \tau$ such that $T_i \cap T \neq 0$, $|T_i \cap T| = 1$ and $|T_i|$ is even.

The necessary condition is not sufficient as the following example shows. Let n and k be both even or both odd and let (I_n, m) be any symmetric quota game with $m(k+1) > (k+1)/n$ and $m(i) \leq i/n$ for $i \leq k$. It is easy to see that in a symmetric quota game, $q_i = 1/n$, so $m(k+1) > (k+1)q_i$. Thus, the game satisfies only the necessary condition, and by theorem 1 it is k-unstable.

<u>Corollary 1.</u> <u>The necessary condition is sufficient if either n is odd and k is even, or if n is even and k is odd, or if $k > (n-2)/2$.</u>

Proof. If n is odd and k is even and T is any set such that $|T| = k+1$, then $|T|$ is odd and $|-T|$ is even, so $m(-T) \geq \sum_{i \epsilon -T} q_i$. It therefore follows that

$$m(T) \leq 1-m(-T) \leq 1 - \sum_{i \epsilon -T} q_i = \sum_{i \epsilon T} q_i,$$

and so the sufficient condition holds. Essentially the same argument applies if n is even and k is odd.

Suppose equation 1 does not hold for T where $|T| = k + 1$. In this case we know that for any $T_i \epsilon \tau$ such that $T_i \cap T \neq 0$, $|T_i \cap T| = 1$ and $|T_i|$ is even, thus there are at least k+1 disjoint sets each having at least two elements, so $n \geq 2(k+1)$, or $k \leq (n-2)/2$. Thus, if $k > (n-2)/2$, the necessary condition is also sufficient.

<u>Corollary 2.</u> <u>Any quota game with an odd number of players is 2-stable.</u>

Proof. The first part of Corollary 1.

II. ON THE STRUCTURE OF SIMPLE QUOTA GAMES AND OF TWO CLOSELY RELATED CLASSES OF GAMES

5. Simple Quota Games

The definition of von Neumann and Morgenstern (4) for simple constant-sum games was extended in (1) to general games as follows: a game is simple if for every $S \subset I_n$, $m(S) = 0$ or 1. The coalitions S with $m(S) = 1$ are called winning. It is trivial that any subset of a losing coalition is losing, that any superset of a winning coalition is winning, and that the complement of a winning coalition is losing. The complement of every losing coalition is winning if and only if the game is constant-sum.

The principal stability result concerning simple games is theorem 4 of (1). It states that a necessary and sufficient condition that a simple game be k-stable is that either there are no (k+1)-element winning coalitions or, if there are, then the intersection of all of them is non-empty. We shall return to this result subsequently, but first we propose to describe the structure of those games which are both simple and quota.

Theorem 5. Let (I_n, m) be a simple game. It is non-constant-sum and quota if and only if either:

i. it is the 4-person game with the 0,1-reduced form characteristic function

$$m(\{i,j\}) = 1, \text{ for } i,j \neq k,$$
$$m(\{i,k\}) = 0, \text{ for } i \neq k$$

(this game will be called the exceptional simple quota game),

or ii. there exists an element k such that any coalition properly including k is winning, and all other coalitions are losing.

<u>It is constant-sum and quota if and only if there exists an element</u> k <u>such that</u> $-\{k\}$ <u>is winning</u>, <u>any coalition properly including</u> k <u>is winning</u>, <u>and all other coalitions are losing</u>.

Proof. It is not difficult to see that the simple games so defined are quota games by taking $q_k=1$, $q_i=0$, $i\neq k$, in the non-exceptional cases and by letting $q_k=-1/2$, $q_i=1/2$, $i\neq k$, in the exceptional 4-person case.

Conversely, suppose (I_n,m) is both a simple and a quota game. Suppose there exists a weak player, which without loss of generality we may take to be n. It is clear that for $i\neq n$, $q_i\geq 0$ and that there exists some $k\neq n$ such that $q_k>0$. For any $i\neq k,n$, $m(\{i,k\}) = q_i+q_k>0$, so $\{i,k\}$ is winning. If we suppose that, in addition to k, there is a j with $q_j>0$, then any set $\{l,j\}$ must also be winning. If $n\geq 5$, then we may choose i,j,k,l all different and different from n such that both $\{i,k\}$ and $\{l,j\}$ are winning, but this is impossible. Thus, if there is a weak player then either $n\leq 4$ or $q_k=1$, $q_i=0$, for $i\neq k,n$. In the latter case, $q_n = 1 - \sum_{j\neq n} q_j = 0$, which contradicts the assumption that n is weak. For $n=4$, the same argument applies as above except if $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ are all winning. In this case, $q_1 + q_2 = q_1 + q_3 = q_2 + q_3 = 1$, so $q_1 = q_2 = q_3 = 1/2$ and $q_4 = -1/2$. Thus $\{i,4\}$, $i\neq 4$, are losing coalitions. For $n=3$, the fact that $\{1,2\}$ is winning implies $q_1 + q_2 = 1$, which implies $q_3=0$, and so there is no weak player.

We may now suppose the game has no weak player. By a repetition of the first argument of the proof we may show that there exists an element k such that any coalition properly including k is winning. If T is any coalition not including k and if $|T| \leq n-2$, then there exists $j\epsilon-(T\cup\{k\})$. Since $\{k,j\}$ is winning and $-T\supset\{k,j\}$, $-T$ is winning and so T is losing.

The only remaining coalition is $-\{k\}$ which if it is losing results in a non-constant-sum game and if it is winning results in a constant-sum game.

Corollary 1. Every constant-sum simple quota game is k-stable for all $k < n-2$ and it is $(n-2)$-unstable. Every non-exceptional non-constant-sum simple quota game is k-stable for all $k \leq n-2$. The exceptional game is 1-unstable.

Proof. Theorem 5 and the conditions for the k-stability of simple games (Theorem 4, [1]).

Corollary 2. The only k-stable pair of a non-exceptional simple quota game is (Q, Δ_n).

Proof. Let (X, τ) be k-stable. By Theorem 3 we know that $X=Q$ except possibly when $k=1$ and n is even. In the latter case, either $\tau=\Delta_n$, in which case it is obvious that $X=Q$, or $\tau=(\{1\},\{2\},\ldots,\{s\},T)$, where T is winning. Obviously $i \in T$, where i is the element described in Theorem 5. If $x_i<1=q_i$, then since $\{i,j\}$, $j \in -T$, is a 1-critical coalition,

$$m(\{i,j\}) = 1 \leq x_i + x_j$$

which implies $x_j>0$ since $x_i<1$. Thus

$$\sum_{r \in T} x_r = 1 - \sum_{r \in -T} x_r < 1 = m(T),$$

which is impossible. Thus $X=Q$. Since $q_i=1$ and $q_j=0$, $j \neq i$, it follows by the second condition of k-stability that $\tau=\Delta_n$.

The case of the non-exceptional non-constant-sum simple quota game is interesting in that it has the property that a set is winning if and only if it properly contains a single element. Obviously, this can be generalized to the case that a coalition is winning if and only if it contains a given non-empty set. We shall characterise such games in section 7, but to do this we must first introduce another concept.

6. The Direct Product of Games

We shall say that a game (I_n, m) is the direct product of games over T and -T, where T is a proper non-empty subset of I_n, if there exist games (T, m') and $(-T, m'')$ such that

$$m(S) = m'(S \cap T)m''(S-T). \tag{2}$$

We observe, first, that given two such games, their composition according to equation 2 yields a game which is the direct product of the given games. This will be shown if we can show that the (I_n, m) which results is in fact a game, i.e., if m is a characteristic function in 0,1-reduced form. This we show:

$$m(I_n) = m(T \cup [-T]) = m'(T)m''(-T) = 1,$$
$$m(\phi) = m'(\phi)m''(\phi) = 0,$$
$$m(\{i\}) = m'(\{i\} \cap T)m''(\{i\}-T) = 0,$$

if R and S are disjoint subsets of I_n,

$$\begin{aligned} m(R \cup S) &= m'([R \cup S] \cap T)m''([R \cup S]-T) \\ &= m'([R \cap T] \cup [S \cap T])m''([R-T] \cup [S-T]) \\ &\geq [m'(R \cap T) + m'(S \cap T)][m''(R-T) + m''(S-T)] \\ &\geq m'(R \cap T)m''(R-T) + m'(S \cap T)m''(S-T) \\ &= m(R) + m(S). \end{aligned}$$

Second, it is clear that if (I_n, m) is the direct product of games over T and -T, the characteristic functions m' and m'' are uniquely determined, namely

$$m'(S) = m(S \cup [-T]), \text{ for } S \subset T,$$

and

$$m''(S) = m(S \cup T), \text{ for } S \subset -T.$$

It should be noted that this concept somewhat parallels that of a decomposable game (3,4). A game (I_n, m) is decomposable into games on T and -T if for every $S \subset I_n$

$$m(S) = m(S \cap T) + m(S-T).$$

Theorem 6. Let (I_n, m) be the direct product of games (T, m') and $(-T, m'')$, then

i. if $R \supset T$ and $S \supset -T$, $m(R \cap S) = m(R)m(S)$;

ii. if $R, S \subset -T$ and $R \cap S = \phi$, then $m(R \cup S \cup T) \geq m(R \cup T) + m(S \cup T)$, and a similar statement if $R, S \subset T$;

iii. $m(T \cup \{i\}) = m(T \cup \{i\}) = 0$ for $i \varepsilon I_n$;

iv. $m(\{i,j,k\}) = 0$, $i,j,k \varepsilon I_n$;

v. (I_n, m) is 1- and 2-stable;

vi. (I_n, m) is non-constant-sum;

vii. (I_n, m) is simple if both (T, m') and $(-T, m'')$ are simple;

viii. (I_n, m) is negative if both (T, m') and $(-T, m'')$ are negative.

Any game with a set T such that i, ii, and iii hold is the direct product of games over T and -T.

Proof. i. Let $Q = R - T$ and $Q' = S \cap T$, then $R \cap S = (T \cup Q) \cap (-T \cup Q') = Q \cup Q'$. Thus, $m(R \cap S) = m'(R \cap S \cap T)m''(R \cap S \cap [-T]) = m'(Q')m''(Q)$. Also, $m(R) = m'(R \cap T)m''(R - T) = m'(T)m''(Q) = m''(Q)$, and $m(S) = m'(S \cap T)m''(S - T) = m'(Q')m''(-T) = m'(Q')$. Thus, $m(R \cap S) = m'(Q')m''(Q) = m(R)m(S)$.

ii. $m(R \cup S \cup T) = m''(R \cup S) \geq m''(R) + m''(S) = m(R \cup T) + m(S \cup T)$.

iii. $m(T \cup \{i\}) = m'(T)m''(\{i\} - T) = 1 \cdot 0 = 0$; similarly $m(-T \cup \{i\}) = 0$.

iv. For any i,j,k either $\{i,j,k\} \subset T \cup \{l\}$ or $\subset -T \cup \{l\}$, where $l = i, j$, or k, and so by iii, $m(\{i,j,k\}) = 0$.

v. Since $m(\{i,j\}) = 0 = m(\{i,j,k\})$, by iv, it is clear that (X, Δ_h) is a 2-stable pair for any distribution X.

vi. Since $m(T) + m(-T) = 0$, by iii, the game is not constant-sum.

vii. Obvious.

viii. If the component games are negative, then for any S

$$m(S) = m'(S \cap T)m''(S - T) \leq \frac{|S \cap T|}{|T|} \frac{|S - T|}{|-T|}.$$

Observe that since $|S\cap T| \le |T|$ and $|S-T| \le |-T|$,

$$|S\cap T|\ |T|(|S-T|-|-T|) \le 0 \le |S-T|\ |-T|(|T|-|S\cap T|),$$

so
$$|S-T|\ |S\cap T|(|-T|+|T|) \le |-T|\ |T|(|S-T|+|S\cap T|),$$

or
$$\frac{|S\cap T|}{|T|}\ \frac{|S-T|}{|-T|} \le \frac{|S-T|+|S\cap T|}{|-T|+|T|} = \frac{|S|}{n}\text{ ; hence } m(S) \le |S|/n.$$

Suppose that i, ii, and iii are met in a game. For $R\subset T$, define $m'(R)=m(R\cup[-T])$ and for $S\subset -T$, $m''(S)=m(S\cup T)$. First, m' and m'' are characteristic functions:

$$m'(T) = m(T\cup[-T]) = 1,$$

$$m'(\phi) = m(-T) = 0, \text{ by iii},$$

$$m'(\{i\}) = m(-T\cup\{i\}) = 0, \text{ by iii},$$

and if $R,S\subset T$ and $R\cap S = \phi$, then using ii we have

$$m'(R\cup S) = m(R\cup S\cup[-T]) \ge m(R\cup[-T]) + m(S\cup[-T]) = m'(R)+m'(S).$$

The direct product of these two games is the given game, for

$$\begin{aligned} m'(S\cap T)m''(S-T) &= m([S\cap T]\cup[-T])m([S-T]\cup T) \\ &= m(S\cup[-T])m(S\cup T) \\ &= m([S\cup(-T)]\cap[S\cup T]) = m(S) \text{ by condition i.} \end{aligned}$$

7. The Structure of Two Classes of Simple Games

Following the lead of section 5 we shall now define three classes of simple games, and we shall determine the structure of two of them. Let (I_n,m) be a simple game, then we shall say that it has:

Property A if there exists a coalition C such that a coalition is winning if and only if it properly contains C;

Property B if there exists a coalition C such that a coalition is winning if and only if it contains C;

Property C if the intersection W of all winning coalitions is non-empty.

It is not difficult to see that if a game has property A, then it has property C where the coalition C is losing and C=W. It is also easy to show that a game has property B if and only if it has property C with W winning. In this case C=W.

From the results on the stability of simple games quoted in section 5, we see that any simple game satisfying one of these properties is k-stable for all k, $0 \leq k \leq n-2$.

Two extreme cases are of interest. If a game has property A and if $|C| = 1$, then by theorem 5 we know that it is a non-exceptional non-constant-sum simple quota game, and conversely. If it has property B and if $|C| = n$, then there is no other winning coalition than I_n, and so for obvious reasons we call this unique game <u>almost inessential</u>. It is not difficult to show that the following conditions are equivalent: a game is almost inessential, it is simple and negative, it is symmetric and the intersection of all winning coalitions is winning, it is negative and the intersection of all winning coalitions is winning.

<u>Theorem 7</u>. <u>Let</u> (I_n,m) <u>be a simple game</u>. <u>The following are equivalent:</u>

i. <u>it has property A and</u> $|C| > 1$,

ii. <u>it is the direct product of an almost inessential game and a non-exceptional non-constant-sum simple quota game</u>.

<u>The following are equivalent:</u>

i. <u>it has property B</u>,

ii. <u>it is decomposable into an almost inessential game and an inessential game</u>,

iii. $m(R \cap S) = m(R)m(S)$ <u>for all</u> $R,S \subset I_n$.

Proof. Suppose the game has property A and $|C| > 1$. Let $i \varepsilon C$ and define $T = C - \{i\}$. We now define games (T,m') and $(-T,m'')$:

$$m'(R) = m(R \cup [-T]), \text{ for } R \subset T,$$

$$m''(S) = m(S \cup T), \text{ for } S \subset -T.$$

In the first case, $m(R\cup[-T]) = 1$ if and only if $R\cup[-T] = R\cup\{i\}\cup[-C]$ properly contains C, which obtains if and only if $R\cup\{i\} = C$, i.e., if and only if R = T. Thus (T, m') is almost inessential. In the second case, $m(S\cup T) = 1$ if and only if $S\cup T = S\cup[C-\{i\}]$ properly contains C, which obtains if and only if S properly includes i. Thus, by theorem 5, $(-T, m'')$ is a non-exceptional non-constant-sum simple quota game. (I_n, m) is, in fact, the direct product of those games since $m(S)=1$ if and only if S properly includes C, in which case $S\cap T=T$ and S-T properly includes i, so $m'(S\cap T) = 1$ and $m''(S-T) = 1$. The converse is obvious.

If the game has property B, then C=W is winning. S is winning if and only if it contains W and so the game is decomposable along W and -W. It is clear that the game on W is almost inessential and that on -W is inessential. The converse is trivial.

Next, if the game has property B, $R\cap S$ is losing if either R or S is losing and it is winning if and only if both R and S are winning, so $m(R\cap S) = m(R)m(S)$. Conversely, if $m(R\cap S) = m(R)m(S)$, then by taking R=S we see that $m(R)=m(R)^2$, so the game must be simple. Let W be the intersection of all winning coalitions. $W\neq\phi$ and W is winning for if R and S are winning so is $R\cap S$. Thus property C holds with W winning, so property B holds.

References

1. Luce, R.D., "A definition of stability for n-person games," Annals of Mathematics, 59, 1954, 357-366.

2. Luce, R.D. and Raiffa, H., A survey of the theory of games: part III, n-person games, Technical Report no. 5, Behavioral Models Project, Bureau of Applied Social Research, Columbia, 1954.

3. McKinsey, J.C.C., Introduction to the theory of games, New York, McGraw-Hill, 1952.

4. Neumann, J. von, and Morgenstern, O., Theory of games and economic behavior, Princeton, Princeton University Press, 1947.

5. Shapley, L.S., "Quota solutions of n-person games," Contributions to the theory of games, II, Annals of Mathematics Studies, no. 28, Princeton, Princeton University Press, 1953, 343-359.